Second-Order Contact Kinematics Between Three-Dimensional Rigid Bodies

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*In this discussion, we provide corrections to the second-order kinematic equations describing contact between three-dimensional rigid bodies, originally published in Sarkar et al. (1996) ["Velocity and Acceleration Analysis of Contact Between Three-Dimensional Rigid Bodies," ASME J. Appl. Mech., **63**(4), pp. 974–984].*

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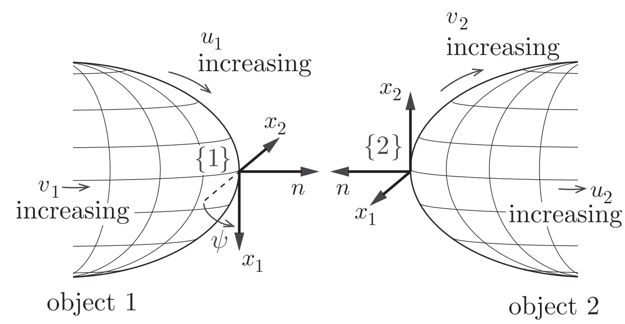

Fig. 1 Objects 1 and 2 are in contact, but they are shown separated for clarity. The surfaces of objects 1 and 2 are parametrized by (u_1, v_1) and (u_2, v_2), respectively. At the point of contact, the unit x_1- and x_2-axes of the coordinate frame {*i*} are in the direction of increasing u_i (and constant v_i) and increasing v_i (and constant u_i), respectively, and the contact normal *n* is the cross product of x_1 and x_2. Rotating frame {2} by ψ about the *n*-axis of frame {1} brings the x_1-axis of the frames {1} and {2} into alignment.

1 Introduction

When a three-dimensional rigid body (object 1) is in single-point contact with another rigid body (object 2), the configuration of object 1 relative to object 2 is five dimensional: the six degrees of freedom of object 1 subject to the single constraint that the distance to object 2 is zero. This five-dimensional configuration space can be parametrized by the two coordinates $U_1=(u_1, v_1)$ describing the contact location on the surface of object 1, the two coordinates $U_2=(u_2, v_2)$ describing the contact location on the surface of object 2, and one coordinate ψ describing the angle of "spin" between frames fixed to each body at the contact point. Collectively, the contact configuration is written as $q=(u_1, v_1, u_2, v_2, \psi)$ (see Fig. 1).

"Contact kinematics" refers to equations relating the relative motion between the objects to the evolution of q. The velocity of one object relative to the other can be written in terms of the linear velocity $V=(V_x, V_y, V_z)$ and the angular velocity $\omega=(\omega_x, \omega_y, \omega_z)$ at a frame at the current contact, where $V_z=0$ is required to maintain contact. The "first-order" contact equations relate (V, ω) (where $V_z=0$) to $\dot{q}$. The "second-order" contact equations express $\ddot{q}$ in terms of the relative linear and angular accelerations (a, α), given an initial state where the first-order contact condition $V_z=0$ is satisfied and a choice of a in the two-dimensional space of linear accelerations that maintain the contact.

These first- and second-order contact kinematics are fundamental to planning and control of robot motions in contact. Such motions are sometimes called "roll-slide" motions. The specialization of these equations to the case of no sliding is useful for manipulation tasks involving rolling.

Second-order contact equations were first published in Ref. [1], based on the work in Sarkar's Ph.D. thesis [2]. These equations generalized Montana's first-order contact kinematics in Ref. [3] and the partial results on second-order contact kinematics in Ref. [4]. Sarkar et al. then restated the second-order contact kinematics in Refs. [5,6], where they were used in the context of robotic manipulation.

Each of the statements of the second-order contact kinematics in Refs. [1,2,5,6] is slightly different, but each contains errors, including sign inversions. Other than a journal typesetting error, the most correct version of the equations is in Ref. [1], which contains only the sign inversions. Given the importance of these equations to robot motion planning and manipulation (the papers [1,2,5,6] have been cited hundreds of times according to Google Scholar) and our own work, we present the corrected equations.

2 Problem Statement

The contact coordinates for each object $i \in \{1, 2\}$ are parametrized by $f_i : U_i \rightarrow \mathbb{R}^3 : (u_i, v_i) \mapsto (x_i, y_i, z_i)$ expressed in a frame fixed to the body. It is assumed that f_i is continuous up to the third derivative (class C^3) so that the local contact geometry (contact frames associated with the first derivative of f_i, curvature associated with the second derivative, and derivative of the curvature associated with the third derivative) is uniquely defined. For details on other definitions, see Ref. [1].

With these definitions, the problem can be stated as follows: given the current state (the relative configurations of the objects and their relative velocity (V, ω) satisfying the first-order contact condition) and their relative acceleration (a, α) satisfying the second-order contact condition, find the contact accelerations $\ddot{q}=(\ddot{u}_1, \ddot{v}_1, \ddot{u}_2, \ddot{v}_2, \ddot{\psi})$.

3 Second-Order Contact Kinematics Derivation

Equations (1), (2), and (3) correspond to Eqs. (39), (41), and (42), respectively, in Ref. [1]. We have rederived and verified these equations, which represent five equality constraints relating the evolution of $\ddot{q}$ and (a, α) when a_z satisfies the second-order condition for maintaining contact given by Eq. (45) in Ref. [1].

$$\sqrt{G_2}(\ddot{U}_2+\bar{\Gamma}_2 W_2) = R_\psi \sqrt{G_1}(\ddot{U}_1+\bar{\Gamma}_1 W_1) + 2\omega_z E_1 R_\psi \sqrt{G_1}\dot{U}_1 + \begin{bmatrix} a_x \\ a_y \end{bmatrix} \quad (1)$$

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$$
R_\psi E_1(\sqrt{G_1})^{-1}(\bar{\bar{L}}_1 W_1 - L_1 \ddot{U}_1) - R_\psi(\sqrt{G_1})^{-1} L_1 \dot{U}_1 \omega_z + \sigma_1 \Gamma_1 \dot{U}_1 \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} = (\sqrt{G_2})^{-1} L_2 \dot{U}_2 \dot{\psi} + E_1(\sqrt{G_2})^{-1}(\bar{\bar{L}}_2 W_2 - L_2 \ddot{U}_2) \quad (2)
$$

$$
-\sigma_1(\Gamma_1 \ddot{U}_1 + \bar{\bar{\Gamma}}_1 W_1) - \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}^T R_\psi E_1 (\sqrt{G_1})^{-1} L_1 \dot{U}_1 + \alpha_z = -\ddot{\psi} + \sigma_2(\Gamma_2 \ddot{U}_2 + \bar{\bar{\Gamma}}_2 W_2) \quad (3)
$$

In these equations, G_i is the metric tensor of object i, $\sigma_i = \sqrt{g_{22,i}/g_{11,i}}$ where $g_{11,i}$ and $g_{22,i}$ are the diagonal entries of G_i, and W_i comprises the velocity product terms $[\dot{u}_i^2, \dot{u}_i\dot{v}_i, \dot{v}_i^2]^T$. The matrices $\Gamma_i, L_i, \bar{\Gamma}_i, \bar{L}_i, \bar{\bar{\Gamma}}_i$, and $\bar{\bar{L}}_i$ describe the local contact geometry as derived from f_i in Ref. [1], and the matrices E_1 and R_ψ are defined as

$$
E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}
$$

The corrected derivation of the contact kinematics begins here. Rearranging Eqs. (1) and (2) yields

$$
R_\psi \sqrt{G_1} \ddot{U}_1 - \sqrt{G_2} \ddot{U}_2 = \sqrt{G_2} \bar{\Gamma}_2 W_2 - R_\psi \sqrt{G_1} \bar{\Gamma}_1 W_1 - 2\omega_z E_1 R_\psi \sqrt{G_1} \dot{U}_1 - \begin{bmatrix} a_x \\ a_y \end{bmatrix} \quad (4)
$$

$$
\begin{aligned}
R_\psi E_1 \left(\sqrt{G_1}\right)^{-1} L_1 \ddot{U}_1 - E_1 \left(\sqrt{G_2}\right)^{-1} L_2 \ddot{U}_2 &= R_\psi E_1 \left(\sqrt{G_1}\right)^{-1} \bar{\bar{L}}_1 W_1 - R_\psi \left(\sqrt{G_1}\right)^{-1} L_1 \dot{U}_1 \omega_z \\
&+ \sigma_1 \Gamma_1 \dot{U}_1 \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} - \left(\sqrt{G_2}\right)^{-1} L_2 \dot{U}_2 \dot{\psi} \\
&- E_1 \left(\sqrt{G_2}\right)^{-1} \bar{\bar{L}}_2 W_2
\end{aligned} \quad (5)
$$

Combining Eqs. (4) and (5) into a single equation yields

$$
\begin{aligned}
\begin{bmatrix} R_\psi \sqrt{G_1} & -\sqrt{G_2} \\ R_\psi E_1 (\sqrt{G_1})^{-1} L_1 & -E_1 (\sqrt{G_2})^{-1} L_2 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix}
&= \begin{bmatrix} -R_\psi \sqrt{G_1} \bar{\Gamma}_1 \\ R_\psi E_1 (\sqrt{G_1})^{-1} \bar{\bar{L}}_1 \end{bmatrix} W_1 + \begin{bmatrix} \sqrt{G_2} \bar{\Gamma}_2 \\ -E_1 (\sqrt{G_2})^{-1} \bar{\bar{L}}_2 \end{bmatrix} W_2 \\
&+ \begin{bmatrix} -2\omega_z E_1 R_\psi \sqrt{G_1} & 0_{2\times 2} \\ -\omega_z R_\psi (\sqrt{G_1})^{-1} L_1 & -(\sqrt{G_2})^{-1} L_2 \dot{\psi} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} \\
&- \begin{bmatrix} 0_{2\times 1} \\ \sigma_1 \Gamma_1 \dot{U}_1 \begin{pmatrix} \omega_y \\ -\omega_x \end{pmatrix} \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} \\ \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} \end{bmatrix} - \begin{bmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} \\ 0_{2\times 1} \end{bmatrix}
\end{aligned} \quad (6)
$$

Following Ref. [1], we define $H_i = (\sqrt{G_i})^{-1} L_i (\sqrt{G_i})^{-1}$, substitute into Eq. (6), and rearrange to get

$$
\begin{aligned}
\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} &= \begin{bmatrix} R_\psi \sqrt{G_1} & -\sqrt{G_2} \\ R_\psi E_1 H_1 \sqrt{G_1} & -E_1 H_2 \sqrt{G_2} \end{bmatrix}^{-1} \\
&\left\{ \begin{bmatrix} \boxed{-R_\psi \sqrt{G_1} \bar{\Gamma}_1} \\ R_\psi E_1 (\sqrt{G_1})^{-1} \bar{\bar{L}}_1 \end{bmatrix} W_1 + \begin{bmatrix} \sqrt{G_2} \bar{\Gamma}_2 \\ \boxed{-E_1 (\sqrt{G_2})^{-1} \bar{\bar{L}}_2} \end{bmatrix} W_2 \right. \\
&+ \begin{bmatrix} -2\omega_z E_1 R_\psi \sqrt{G_1} & 0_{2\times 2} \\ -\omega_z R_\psi H_1 \sqrt{G_1} & -\dot{\psi} H_2 \sqrt{G_2} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} \\
&\left. - \begin{bmatrix} 0_{2\times 1} \\ \sigma_1 \Gamma_1 \dot{U}_1 \begin{pmatrix} \omega_y \\ -\omega_x \end{pmatrix} \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} \\ \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} \end{bmatrix} - \begin{bmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} \\ 0_{2\times 1} \end{bmatrix} \right\}
\end{aligned} \quad (7)
$$

This is equivalent to Eq. (43) of Ref. [1] except the two boxed terms are preceded by a minus sign in the corrected equations. The fifth equation for $\ddot{\psi}$ is derived by rearranging Eq. (3)

$$
\ddot{\psi} = \boxed{-\begin{bmatrix} \omega_y \\ -\omega_x \end{bmatrix}^T R_\psi E_1 (\sqrt{G_1})^{-1} L_1 \dot{U}_1} - \alpha_z + \sigma_1(\Gamma_1 \ddot{U}_1 + \bar{\bar{\Gamma}}_1 W_1) + \sigma_2(\Gamma_2 \ddot{U}_2 + \bar{\bar{\Gamma}}_2 W_2) \quad (8)
$$

where the minus sign on the left of the boxed term did not appear in Eq. (44) in Ref. [1].

4 Example: A Sphere Rolling on a Sphere

Consider the example of a small sphere (object 1) rolling without sliding on a larger sphere (object 2), as shown in Fig. 2. The spheres $i \in \{1, 2\}$ are parameterized by

$$
f_i : U_i \to \mathbb{R}^3 : (u_i, v_i) \mapsto (\rho_i \sin(u_i)\cos(v_i), \rho_i \sin(u_i)\sin(v_i), \rho_i \cos(u_i)) \quad (9)
$$

where the "latitude" u_i satisfies $0 < u_i < \pi$ and the "longitude" v_i satisfies $-\pi < v_i < \pi$.

Object 2 (the large red sphere) remains fixed in space while object 1 (the small blue sphere) rolls on it. To ensure rolling, the relative linear velocity at the contact satisfies $V = 0$ and the linear acceleration a satisfies the three constraints given in Eq. (60) of Ref. [1]: one constraint to maintain contact and two constraints that prevent slip. Relative velocities and accelerations are defined as the motion of the contact frame on object 1 relative to the contact frame on object 2.

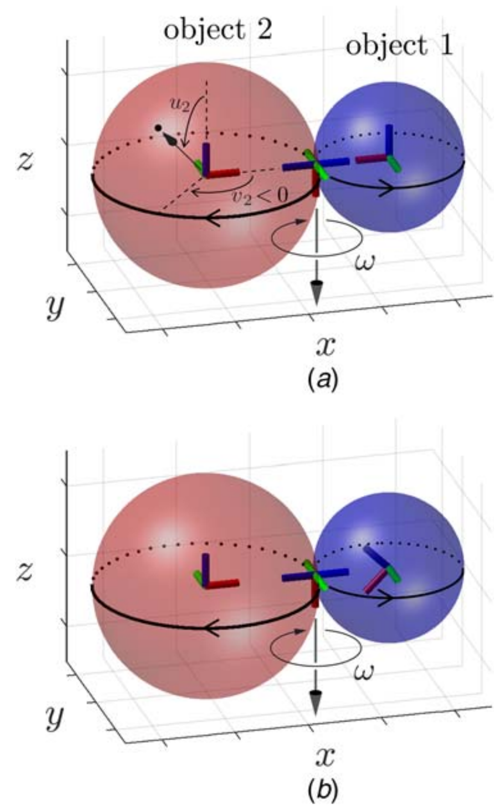

Fig. 2 Small blue sphere: the rolling object 1. Large red sphere: the stationary object 2. From both initial configurations, object 1 (the small sphere) is made to roll on the equator of object 2 (the large sphere) by rotating about the downward-pointing axis in the contact tangent plane at all times. (*a*) Initial configuration $q_o = (\pi/2, 0, \pi/2, 0, 0)$. The (u_2, v_2) representation of an example point on the surface of object 2 is shown. (*b*) Initial configuration $q_0 = (\pi/4, 0, \pi/2, 0, 0)$.

Object 1 is made to roll at a constant speed along the "equator" of object 2 by choosing $(\omega_x, \omega_y, \omega_z) = (\omega, 0, 0)$: object 1 always rotates about the downward-pointing x_1-axis of frame {2} in the tangent plane of the contact.

4.1 Instantaneous Solution. Consider the case where the initial configuration is $q_0 = (\pi/2, 0, \pi/2, 0, 0)$ as shown in Fig. 2(*a*). From the first-order kinematics in Ref. [1], we can solve for the initial contact velocities

$$\dot{q}_0 = (0, \rho_2\omega k_1, 0, -\rho_1\omega k_1, 0) \tag{10}$$

where $k_1 = 1/(\rho_1 + \rho_2)$. Using the initial contact state $(q_0, \dot{q}_0)$, the controls $(\alpha_x, \alpha_y, \alpha_z) = (0, 0, 0)$, and the rolling assumptions $(v_x, v_y, v_z) = (0, 0, 0)$, we solve for (a_x, a_y, a_z) to satisfy the second-order rolling conditions (Eq. (60) of Ref. [1]) and the initial coordinate acceleration $\ddot{q}_0$. For both the original equations (43) and (44) in Ref. [1] and the corrected equations (7) and (8), we get

$$\ddot{q}_0 = (\ddot{u}_1, \ddot{v}_1, \ddot{u}_2, \ddot{v}_2, \ddot{\psi}) = (0, 0, 0, 0, 0) \tag{11}$$

The v_1 and v_2 coordinates change linearly with time while all other contact coordinates remain constant, as would be expected for rolling along the equators. The original contact kinematic equations give correct answers when the boxed terms in Eqs. (7) and (8) are zero.

For the initial configuration in Fig. 2(*b*), however, everything is the same except object 1 is tilted by $\pi/4$, i.e., the initial configuration is $q_0 = (\pi/4, 0, \pi/2, 0, 0)$. According to the first-order kinematics in Ref. [1]

$$\dot{q}_0 = (0, \sqrt{2}\rho_2\omega k_1, 0, -\rho_1\omega k_1, \rho_2\omega k_1) \tag{12}$$

where $k_1 = 1/(\rho_1 + \rho_2)$. Solving the corrected equations (7) and (8), we obtain

$$\ddot{q}_0 = (\ddot{u}_1, \ddot{v}_1, \ddot{u}_2, \ddot{v}_2, \ddot{\psi}) = (k_2, 0, 0, 0, 0) \tag{13}$$

where $k_2 = \rho_2^2\omega^2/(\rho_1 + \rho_2)^2$. As expected, u_2 remains constant (contact remains on the "equator" of object 2) as v_2 (the "longitude") changes with time. On the other hand, Eqs. (43) and (44) in Ref. [1] yield

$$\ddot{q}_0 = (\ddot{u}_1, \ddot{v}_1, \ddot{u}_2, \ddot{v}_2, \ddot{\psi}) = \left(\left(1 - \frac{2\rho_1}{\rho_1 + \rho_2}\right)k_2, 0, \left(\frac{2\rho_1}{\rho_1 + \rho_2}\right)k_2, 0, 0\right) \tag{14}$$

The nonzero value of $\ddot{u}_2$ shows that the contact point incorrectly accelerates away from the equator of object 2.

4.2 Simulation. The errors in the original equations are clearly demonstrated by simulation. Figure 3 shows the results of numerical simulations of the original second-order kinematics from Ref. [1] and the corrected equations in this paper for $\omega = 2.5$, sphere radii $\rho_1 = 2$ and $\rho_2 = 3$, and the initial configurations shown in Fig. 2. A video of these simulations is available as Supplemental Material on the ASME Digital Collection.

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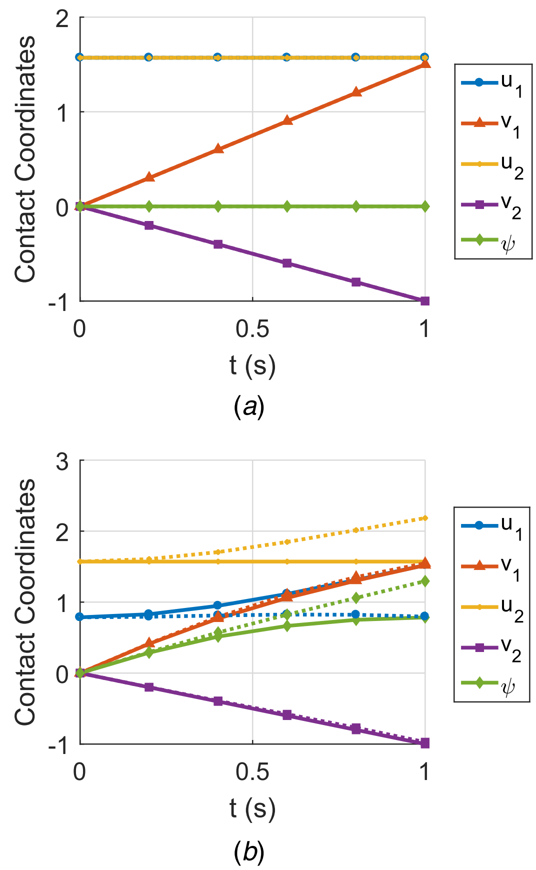

Fig. 3 Simulated second-order rolling trajectories with the original kinematics (Ref. [1]) shown by the dotted lines and the corrected kinematics shown by the solid lines. Note that u_2 and $\dot{v}_2$ should be constant for both (*a*) and (*b*). The original and corrected kinematics yield the same results in (*a*) because the incorrect terms in the original kinematics are zero, just as they are in the corrected kinematics. The errors in the original kinematics become clear in the simulation in (*b*). (*a*) Simulation starting from $q_o = (\pi/2, 0, \pi/2, 0, 0)$. (*b*) Simulation starting from $q_0 = (\pi/4, 0, \pi/2, 0, 0)$.

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